LEECH LATTICE AND HOLOMORPHIC VERTEX OPERATOR ALGEBRAS OF CENTRAL CHARGE 24

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1. Introduction

This manuscript is based on a joint work with Hiroki Shimakura of Tohoku University and my talk at RIMS, University of Kyoto on Decemeber, 2017. We discuss an idea of G. Höhn [Hö], who proposed a construction of holomorphic vertex operator algebras of central charge 24 by using the Leech lattice. In particular, we explain how to prove one of his conjectures.

In some sense, Höhn's idea is a generalization of the theory of Cartan subalgebras to VOA. The main idea is to study the commutant of the subVOA generated by a Cartan subalgebra of V_1. Now let us explain his ideas in detail. Let $\mathfrak{g}$ be a Lie algebra in Schellekens' list [Sc93]. Suppose that $\mathfrak{g}$ is semisimple and let

$$\mathfrak{g} = \mathfrak{g}_{1,k_1} \oplus \cdots \oplus \mathfrak{g}_{r,k_r},$$

where $\mathfrak{g}_{i,k_i}$'s are simple ideals of $\mathfrak{g}$ at level k_i. Let V be a strongly regular holomorphic VOA of central charge 24 such that $V_1 \cong \mathfrak{g}$. Then the subVOA U generated by V_1 is isomorphic to the tensor of simple affine VOAs

$$L_{\widehat{\mathfrak{g}_1}}(k_1, 0) \otimes \cdots \otimes L_{\widehat{\mathfrak{g}_r}}(k_r, 0).$$

and U and V have the same conformal element [DM04b]. For each $1 \leq i \leq r$, the affine VOA $L_{\mathfrak{g}_i}(k_i, 0)$ contains the lattice VOA $V_{\sqrt{k_i}Q_L^i}$, where Q_L^i is the lattice spanned by the long roots of $\mathfrak{g}_i$ (see [DW]).

Let $Q_\mathfrak{g} = \sqrt{k_1}Q_L^i \oplus \cdots \oplus \sqrt{k_r}Q_L^r$ and set $W = \mathrm{Com}_V(V_{Q_\mathfrak{g}})$ and $X = \mathrm{Com}_V(W)$. Then $\mathrm{Com}_V(X) = W$ and $\mathrm{Com}_V(W) = X \supset V_{Q_\mathfrak{g}}$.

Since an extension of a lattice VOA is again a lattice VOA, there exists an even lattice $L_\mathfrak{g} > Q_\mathfrak{g}$ such that $X \cong V_{L_\mathfrak{g}}$. Notice that $V_{L_\mathfrak{g}}$ has group-like fusion, i.e., all irreducible $V_{L_\mathfrak{g}}$-modules are simple current modules [DL93, Corollary 12.10]. In addition, $R(V_{L_\mathfrak{g}})$ is isomorphic to $\mathcal{D}(L_\mathfrak{g}) = L_\mathfrak{g}^*/L_\mathfrak{g}$ as quadratic spaces. Indeed, $R(V_{L_\mathfrak{g}})$ has the quadratic form

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$q : R(V_{L_\mathfrak{g}}) \to \mathbb{Z}/n\mathbb{Z}$ defined by

$$q(V_{\alpha+L_\mathfrak{g}}) = n\mathrm{wt}(V_{\alpha+L_\mathfrak{g}}) = \frac{n\langle \alpha, \alpha\rangle}{2} \mod n,$$

where $\mathrm{wt}(\cdot)$ denotes the conformal weight of the module and n is the smallest integer such that $n\langle \alpha,\alpha\rangle \in 2\mathbb{Z}$ for all $\alpha \in L_\mathfrak{g}^*$.

By [KMa12, Lin17], the VOA W also has group-like fusion and $R(W)$ forms a quadratic space isomorphic to $R(V_{L_\mathfrak{g}})$. In this case, the VOA V defines a maximal totally singular subspace of $R(V_{L_\mathfrak{g}}) \times R(W)$ and it induces an isomorphism $\varphi : (R(V_{L_\mathfrak{g}}), q) \to (R(W), q')$ of quadratic spaces such that $q(M) + q'(\varphi(M)) = 0$ for all $M \in R(V_{L_\mathfrak{g}})$.

Conversely, let $\varphi : (R(V_{L_\mathfrak{g}}), q) \to (R(W), -q')$ be an isomorphism of quadratic spaces. Then the set $\{(M, \varphi(M)) \mid M \in R(V_{L_\mathfrak{g}})\}$ is a maximal totally singular subspace of $R(V_{L_\mathfrak{g}}) \times R(W)$ and $U = \bigoplus_{M\in R(V_{L_\mathfrak{g}})} M \otimes \varphi(M)$ is a holomorphic VOA.

The key idea of Höhn is to try to describe the VOA W using a certain coinvariant sublattice of the Leech lattice Λ. The following is his main conjecture.

Conjecture 1.1. For each semisimple case in Schellekens' list, there exists an isometry $g \in O(\Lambda)$ such that $R(V_{\Lambda_g}^{\hat{g}}) \cong R(V_{L_\mathfrak{g}})$ as quadratic spaces, where Λ_g denotes the coninvariant sublattice of g in Λ (see Definition 2.1).

In [Hö], Höhn also gives some description of the isometry g for each case. In this article, we will sketch a proof for the above conjecture. In fact, a more general statement holds (see Theorems 4.8, 5.1, 5.3 and 5.5).

Main Theorem 1. *Let L be an even unimodular lattice. Let g be an element in $O(L)$ and $\hat{g}$ a lift of g. Then the VOA $V_{L_g}^{\hat{g}}$ has group-like fusion, namely, all irreducible modules of $V_{L_g}^{\hat{g}}$ are simple current modules.*

We will also explain how to determine the quadratic space structure for $(R(V_{L_g}^{\hat{g}}), q)$, where q is defined by the conformal weights modulo $\mathbb{Z}$ (see Section 5). It turns out that $(R(V_{L_g}^{\hat{g}}), q)$ can be determined by the corresponding structure of $(R(V_L^{\phi_g}), q)$ and the decomposition of $V_L^{\phi_g}$ as a sum of irreducible $V_{L_g}^{\hat{g}} \otimes V_{L^g}$-modules, where ϕ_g denotes a standard lift of g in $\mathrm{Aut}\,(V_L)$. There are several different cases which depend on the order of ϕ_g and the conformal weights of the corresponding twisted modules.

2. Preliminary

By *lattice*, we mean a free abelian group of finite rank with a rational valued, positive definite symmetric bilinear form $\langle\ ,\ \rangle$. A lattice L is *integral* if $\langle L, L\rangle < \mathbb{Z}$ and it is *even* if $\langle x, x\rangle \in 2\mathbb{Z}$ for any $x \in L$. Note that an even lattice is integral. Let $L^* = \{v \in \mathbb{Q}\otimes_\mathbb{Z} L \mid$

$\langle v, L\rangle < \mathbb{Z}\}$ be the dual lattice of L. Note that if a lattice L is integral, then $L \subset L^*$ and we denote the discriminant group L^*/L by $\mathcal{D}(L)$. Let $\{x_1, \dots, x_n\}$ be a basis of L. The Gram matrix of L is defined to be the matrix $G = (\langle x_i, x_j\rangle)_{1\le i,j\le n}$. The determinant of L, denoted by $\det(L)$, is the determinant of G. Note that $\det(L) = |\mathcal{D}(L)|$.

Definition 2.1. Let L be an integral lattice and $g \in O(L)$. We denote the fixed point sublattice of g by $L^g = \{x \in L \mid gx = x\}$. The *coinvariant lattice* of g is defined to be

$$L_g = Ann_L(L^g) = \{x \in L \mid \langle x, y\rangle = 0 \text{ for all } y \in L^g\}.$$

First we recall the following simple observation from [LS17].

Lemma 2.2. *Let L be an even unimodular lattice. Let $g \in O(L)$ be an isometry of order $\ell > 1$ such that $L^g \neq 0$. Then $\ell (L^g)^* < L^g$ and $\mathcal{D}(L^g) \cong \mathcal{D}(L_g)$.*

By the above lemma, we have $\ell\lambda \in L_g$ for any $\lambda \in L_g^*$ and hence the exponent of the group L_g^*/L_g divides ℓ.

3. Lattice VOAs, automorphisms and twisted modules

In this section, we review the construction of a lattice VOA and the structure of its automorphism group from [FLM88, DN99]. We also review a construction of irreducible twisted modules for (standard) lifts of isometries from [Le85, DL96] (see also [BK04]).

3.1. **Lattice VOA and the automorphism group.** Let L be an even lattice of rank m and let $(\cdot|\cdot)$ be the positive-definite symmetric bilinear form on $\mathbb{R}\otimes_{\mathbb{Z}} L \cong \mathbb{R}^m$. The lattice VOA V_L associated with L is defined to be $M(1) \otimes \mathbb{C}\{L\}$. Here $M(1)$ is the Heisenberg VOA associated with $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$ and the form $(\cdot|\cdot)$ extended $\mathbb{C}$-bilinearly. That $\mathbb{C}\{L\} = \bigoplus_{\alpha\in L} \mathbb{C}e^\alpha$ is the twisted group algebra with commutator relation $e^\alpha e^\beta = (-1)^{(\alpha|\beta)} e^\beta e^\alpha$, for $\alpha, \beta \in L$. We fix a 2-cocycle $\varepsilon(\cdot|\cdot) : L \times L \to \{\pm 1\}$ for $\mathbb{C}\{L\}$ such that $e^\alpha e^\beta = \varepsilon(\alpha|\beta) e^{\alpha+\beta}$, $\varepsilon(\alpha|\alpha) = (-1)^{(\alpha|\alpha)/2}$ and $\varepsilon(\alpha|0) = \varepsilon(0|\alpha) = 1$ for all $\alpha, \beta \in L$. It is well-known that the lattice VOA V_L is strongly regular, and its central charge is equal to m, the rank of L.

Let $\hat{L}$ be the central extension of L by $\langle -1\rangle$ associated with the 2-cocycle $\varepsilon(\cdot|\cdot)$. Let $\operatorname{Aut}\hat{L}$ be the set of all group automorphisms of $\hat{L}$. For $\varphi \in \operatorname{Aut}\hat{L}$, we define the element $\bar{\varphi} \in \operatorname{Aut} L$ by $\varphi(e^\alpha) \in \{\pm e^{\bar{\varphi}(\alpha)}\}$, $\alpha \in L$. Set

$$O(\hat{L}) = \{\varphi \in \operatorname{Aut}\hat{L} \mid \bar{\varphi} \in O(L)\}.$$

For $\chi \in \operatorname{Hom}(L, \mathbb{Z}_2)$, the map $\hat{L} \to \hat{L}$, $e^\alpha \mapsto (-1)^{\chi(\alpha)} e^\alpha$, is an element in $O(\hat{L})$. Such automorphisms form an elementary abelian 2-subgroup of $O(\hat{L})$ of rank m, which is also denoted by $\mathrm{Hom}(L,\mathbb{Z}_2)$. It was proved in [FLM88, Proposition 5.4.1] that the following sequence is exact:

$$1 \longrightarrow \mathrm{Hom}(L,\mathbb{Z}_2) \longrightarrow O(\hat{L}) \xrightarrow{-} O(L) \longrightarrow 1. \tag{3.1}$$

We identify $O(\hat{L})$ as a subgroup of $\mathrm{Aut}\, V_L$ as follows: for $\varphi \in O(\hat{L})$, the map

$$\alpha_1(-n_1)\dots\alpha_m(-n_s)e^{\beta} \mapsto \bar{\varphi}(\alpha_1)(-n_1)\dots\bar{\varphi}(\alpha_s)(-n_s)\varphi(e^{\beta})$$

is an automorphism of V_L, where $n_1,\dots,n_s \in \mathbb{Z}_{>0}$ and $\alpha_1,\dots,\alpha_s,\beta \in L$.

Let $N(V_L) = \langle \exp(a_{(0)}) \mid a \in (V_L)_1 \rangle$, which is called the *inner automorphism group* of V_L. We often identify $\mathfrak{h}$ with $\mathfrak{h}(-1)\mathbf{1}$ via $h \mapsto h(-1)\mathbf{1}$. For $v \in \mathfrak{h}$, set

$$\sigma_v = \exp(-2\pi\sqrt{-1}v_{(0)}) \in N(V_L).$$

Note that σ_v is the identity map of V_L if and only if $v \in L^*$. Let

$$D = \{\sigma_v \mid v \in \mathfrak{h}/L^*\} \subset N(V_L).$$

Note that $\mathrm{Hom}(L,\mathbb{Z}_2) = \{\sigma_v \mid v \in (L^*/2)/L^*\} \subset D$ and that for $\varphi \in O(\hat{L})$ and $v \in \mathfrak{h}$, we have $\varphi\sigma_v\varphi^{-1} = \sigma_{\bar{\varphi}(v)}$.

Proposition 3.1 ([DN99, Theorem 2.1]). *The automorphism group* $\mathrm{Aut}\, V_L$ *of* V_L *is generated by the normal subgroup* $N(V_L)$ *and the subgroup* $O(\hat{L})$.

3.2. **Lifts of isometries of lattices.**

Definition 3.2 ([Le85] (see also [EMS])). An element $\varphi \in O(\hat{L})$ is called a *lift* of $g \in O(L)$ if $\bar{\varphi} = g$, where the map $^-$ is defined as in (3.1). A lift ϕ_g of $g \in O(L)$ is said to be *standard* if $\phi_g(e^{\alpha}) = e^{\alpha}$ for all $\alpha \in L^g = \{\beta \in L \mid g(\beta) = \beta\}$.

Remark 3.3. For any $g \in O(L)$, a lift $\hat{g} \in O(\hat{L})$ corresponds to a map $u : L \to \{\pm 1\}$ such that $u(\alpha+\beta)/u(\alpha)u(\beta) = \varepsilon(\alpha,\beta)/\varepsilon(g\alpha,g\beta)$ and $\hat{g}(e^{\alpha}) = u(\alpha)e^{g\alpha}$.

Proposition 3.4 ([Le85, Section 5]). *For any isometry of* L*, there exists a standard lift.*

The orders of standard lifts are determined in [Bo92, Lemma 12.1] (cf. [EMS]) as follows:

Lemma 3.5 ([Bo92]). *Let* $g \in O(L)$ *be of order* n *and let* ϕ_g *be a standard lift of* g.

(1) *If* n *is odd, then the order of* ϕ_g *is also* n.
(2) *Assume that* n *is even. Then* $\phi_g^n(e^{\alpha}) = (-1)^{(\alpha|g^{n/2}(\alpha))}e^{\alpha}$ *for all* $\alpha \in L$*. In particular, if* $(\alpha|g^{n/2}(\alpha)) \in 2\mathbb{Z}$ *for all* $\alpha \in L$*, then the order of* ϕ_g *is* n*; otherwise the order of* ϕ_g *is* $2n$.

Remark 3.6 (See [DL96, EMS, LS2]). A standard lift of an isometry is unique, up to conjugation.

3.3. **Irreducible twisted modules for lattice VOAs.** Let $g \in O(L)$ be of order p and $\phi_g \in O(\hat{L})$ a standard lift of g. Let $(L^*/L)^g$ be the set of cosets of L in L^* fixed by g. Then V_L has exactly $|(L^*/L)^g|$ irreducible ϕ_g-twisted V_L-modules, up to isomorphism (see [DLM00]). The irreducible irreducible ϕ_g-twisted V_L-modules have been constructed in [Le85, DL96] explicitly and are classified in [BK04]. They are given by

$$V_{\lambda+L}[\phi_g] = M(1)[g] \otimes \mathbb{C}[P_0^g(\lambda+L)] \otimes T_{\tilde{\lambda}}, \qquad \text{for } \lambda + L \in (L^*/L)^g, \tag{3.2}$$

where $M(1)[g]$ is the "g-twisted" free bosonic space, $\mathbb{C}[\lambda + P_0^g(L)]$ is a module for the group algebra of $P_0^g(L)$ and $T_{\tilde{\lambda}}$ is an irreducible module for a certain "g-twisted" central extension of L_g associated with $\tilde{\lambda} = (1-g)\lambda$. (see [Le85, Propositions 6.1 and 6.2] and [DL96, Remark 4.2] for detail). Recall that $\dim T_{\tilde{\lambda}} = |L_g/(1-g)L|^{1/2}$. Note also that $M(1)[g]$ is spanned by vectors of the form

$$x_1(-m_1)\ldots x_s(-m_s)1,$$

where $m_i \in (1/n)\mathbb{Z}_{>0}$, $x_i \in \mathfrak{h}_{(nm_i)}$ for $1 \le i \le s$, and $\mathfrak{h}_{(j)} = \mathfrak{h}_{(j;g)} = \{x \in \mathfrak{h} \mid g(x) = \exp((j/n)2\pi\sqrt{-1})x\}$.

Then the conformal weight of $x_1(-m_1)\ldots x_s(-m_s) \otimes e^\alpha \otimes t \in V_{\lambda+L}[\phi_g]$ is given by

$$\sum_{i=1}^{s} m_i + \frac{(\alpha|\alpha)}{2} + \rho_T, \tag{3.3}$$

where

$$\rho_T = \frac{1}{4n^2}\sum_{j=1}^{n-1} j(n-j)\dim \mathfrak{h}_{(j)},$$

$x_1(-m_1)\ldots x_s(-m_s) \in M(1)[g]$, $e^\alpha \in \mathbb{C}[P_0^g(\lambda+L)]$ and $t \in T_{\lambda'}$.

3.4. **Non-standard lifts.** For an arbitrary lift $\hat{g}$ of g, we have

$$\hat{g} = \sigma_\mu \phi_g \quad \text{for some } \mu \in \mathfrak{h}_{(0)}.$$

Suppose that the order of $\hat{g}$ is the same as that of ϕ_g. Then $\sigma_\mu^n = 1$ on V_L, where $n = |\phi_g|$, which is either p or $2p$. Therefore, $\mu \in (\frac{1}{n}L^*) \cap \mathfrak{h}_{(0)}$.

We first recall the following result from [Li97].

Proposition 3.7 ([Li97, Proposition 5.4]). *Let g be an automorphism of V of finite order and let $h \in V_1$ such that $\sigma(h) = h$. We also assume $h_{(0)}$ acts semisimply on V and* Spec $h_{(0)} < (1/T)\mathbb{Z}$ *for a positive integer T. Let (M, Y_M) be a g-twisted V-module and define $(M^{(h)}, Y_{M^{(h)}}(\cdot, z))$ as follows:*

$$M^{(h)} = M \quad \textit{as a vector space;}$$

$$Y_{M^{(h)}}(a,z) = Y_M(\Delta(h,z)a, z) \quad \textit{for any } a \in V,$$

where $\Delta(h,z) = z^{h_{(0)}} \exp\left(\sum_{n=1}^{\infty} \frac{h_{(n)}}{-n}(-z)^{-n}\right)$. *Then* $(M^{(h)}, Y_{M^{(h)}}(\cdot,z))$ *is a* $\sigma_h g$*-twisted* V*-module. Furthermore, if* M *is irreducible, then so is* $M^{(h)}$.

By the proposition above, the module $V_{\lambda+L}[\phi_g]^{(\mu)}$ is an irreducible $\hat{g} = \sigma_\mu \phi_g$-twisted V_L-module. The conformal weight of $x_1(-m_1)\dots x_s(-m_s)\otimes e^\alpha \otimes t$ in $V_{\lambda+L}[\phi_g]^{(\mu)}$ ($m_i \in (1/n)\mathbb{Z}_{>0}$, $\alpha \in P_0^g(\lambda+L)$ and $t\in T$) is given by

$$\sum_{i=1}^{s} m_i + \frac{(\mu+\alpha|\mu+\alpha)}{2} + \rho_T. \tag{3.4}$$

As a vector space, we also have

$$V_{\lambda+L}[\phi_g]^{(\mu)} \cong M(1)[g]\otimes \mathbb{C}[\mu+\lambda'+P_0^g(L)]\otimes T_{\tilde{\lambda}} \tag{3.5}$$

For simplicity, we denote it by $V_{\lambda+L}[\hat{g}]$.

4. Orbifold VOA $V_{L_g}^{\hat{g}}$

Let L be an even unimodular lattice and g an isometry of L. In this section, we study the orbifold VOA $V_{L_g}^{\hat{g}}$. In particular, we show that the orbifold VOA $V_{L_g}^{\hat{g}}$ has group-like fusion, i.e., $R(V_{L_g}^{\hat{g}})$ forms an abelian group with respect to the fusion product, or equivalently all irreducible modules of $V_{L_g}^{\hat{g}}$ are simple current modules.

Let V be a VOA and $g\in \mathrm{Aut}(V)$. For any irreducible module M of V, we denote the g-conjugate of M by $M\circ g$, i.e., $M\circ g = M$ as a vector space and the vertex operator $Y_{M\circ g}(u,z) = Y_M(gu,z)$ for $u\in V$.

If $V=V_L$ is a lattice VOA and $\hat{g}$ is a lift of an isometry $g\in O(L)$, then $V_{\alpha+L}\circ\hat{g}\cong V_{g\alpha+L}$ for $\alpha+L\in L^*/L$.

Proposition 4.1 ([DM97, Theorem 6.1]). *Let* V *be a simple VOA and* $g\in\mathrm{Aut}(V)$. *Let* M *be an irreducible module of* V. *Suppose* $M\not\cong M\circ g$. *Then* M *is also irreducible as a* V^g*-module and* $M\cong M\circ g$ *as* V^g*-modules.*

Therefore, if $V_L^{\hat{g}}$ has group-like fusion, then $\alpha+L = g\alpha+L$ for all $\alpha+L\in L^*/L$; otherwise, $\alpha+L\neq g\alpha+L$ for some α and $V_{\alpha+L}\boxtimes_{V_L^{\hat{g}}} V_{\alpha+L}\supset V_{2\alpha+L}+V_{\alpha+g\alpha+L}$. This condition always holds in our setting.

Lemma 4.2. *Let* L *be an even unimodular lattice and* $g\in O(L)$. *Suppose* $g\neq 1$. *Then* $(1-g)L_g^* < L_g$ *and hence* $\alpha+L_g = g\alpha+L_g$ *for all* $\alpha+L_g\in L_g^*/L_g$.

The following lemma follows immediately by the Elementary Factor Theorem and will be used in the computation of quantum dimensions.

Lemma 4.3. *Let L be a lattice. Let $g \in O(L)$ be fixed point free isometry of L. Then we have $|L/(1-g)L| = \det(1-g)$.*

Next we compute the quantum dimensions for irreducible g-twisted modules for V_L.

We first recall some facts about quantum dimensions of irreducible modules of vertex operator algebras from [DJX13]. Let V be a strongly regular VOA and let $M^0 = V, M^1, ..., M^p$ be all the irreducible V-modules. The *quantum dimension* of M^i is defined to be

$$\mathrm{qdim}_V M^i = \lim_{y\to 0} \frac{Z_{M^i}(iy)}{Z_V(iy)},$$

where $Z_M(\tau) = Z_M(\mathbf{1}, \tau)$ is the character of M and y is real and positive.

The following result was proved in [DJX13].

Theorem 4.4. *Let V be a strongly regular vertex operator algebra, $M^0 = V, M^1, ..., M^p$ be all the irreducible V-modules. Assume further that the conformal weights of $M^1, ..., M^p$ are greater than 0. Then*

(1) $\mathrm{qdim}_V M^i \geq 1$ *for any* $0 \leq i \leq p$;

(2) M^i *is a simple current if and only if* $\mathrm{qdim}\, M^i = 1$.

First we assume that g is a fixed point free isometry on L. In this case, $L^g = 0$ and the irreducible ϕ_g-twisted modules are given by $V_L^T = M(1)[g] \otimes T$ (see Section 3.3).

The following result can be found in [ALY].

Theorem 4.5. *Let L be positive definite even lattice of rank ℓ. Let g be a fixed point free isometry of L of order p. Let $\hat{g}$ be a lift of g. For any $\widehat{L}_g$-module T, the quantum dimension of the $\hat{g}$-twisted module V_L^T exists and*

$$\mathrm{qdim}_{V_L} V_L^T := \lim_{q\to 1^-} \frac{Z_{V_L^T}(q)}{Z_{V_L}(q)} = \frac{v \dim T}{\prod_{d|p} d^{m_d/2}},$$

where $v = \sqrt{|\mathcal{D}(L)|}$ and m_d are integers given by $\det(x-g) = \prod_{d|p}(x^d-1)^{m_d}$.

As a corollary, we have

Corollary 4.6. *Let L be an even unimodular lattice and let $g \in O(L)$. Then*

$$\mathrm{qdim}_{V_{L_g}} V_{L_g}^T = 1$$

for any irreducible $(\widehat{L_g})_g$-module T.

Proof. It follows from Theorem 4.5 and Lemma 4.3. □

Now set $N = L_g$. Then $L_{g^i} < N$ and $L_{g^i} = N_{g^i}$ for any i. Let λ' and λ'' be the images of λ under the natural projections from N to $(N^{g^i})^*$ and $(N_{g^i})^*$, respectively. Then

$$V_N = \bigoplus_{\lambda \in N/(N^{g^i} \perp N_{g^i})} V_{\lambda' + N^{g^i}} \otimes V_{\lambda'' + N_{g^i}}.$$

Notice that $\hat{g}^i$ may no longer be a standard lift of g^i and hence $\hat{g}^i = \sigma_\mu \phi_{g^i}$ for some μ. By (3.5), the irreducible twisted module

$$V_{\lambda+L_g}[\hat{g}^i] = M(1)[g] \otimes \mathbb{C}[\mu + \lambda' + P_0^{g^i}(L_g)] \otimes T_{\tilde{\lambda}}.$$

Then as a $\hat{g}^i$-twisted module of $V_{N^{g^i}} \otimes V_{N_{g^i}}$-module, we have

$$V_{\lambda+L_g}[\hat{g}^i] = \bigoplus_{\lambda \in N/(N^{g^i} \perp N_{g^i})} V_{\mu+\lambda'+N^{g^i}} \otimes V_{\lambda''+N_{g^i}}[\hat{g}^i].$$

Moreover,

$$\frac{Z_{V_{\lambda+L_g}[\hat{g}^i]}(iy)}{Z_{V_{L_g}}(iy)} = \frac{\sum_{\lambda \in N/(N^{g^i} \perp N_{g^i})} Z_{V_{\mu+\lambda'+N^{g^i}}}(iy) Z_{V_{\lambda''+N_{g^i}}[\hat{g}^i]}(iy)}{\sum_{\lambda \in N/(N^{g^i} \perp N_{g^i})} Z_{V_{\lambda'+N^{g^i}}}(iy) Z_{V_{\lambda''+N_{g^i}}}(iy)}.$$

By a direct calculation, it is easy to obtain the following lemma.

Lemma 4.7. *Let L be an even unimodular lattice and let $g \in O(L)$. Let $\hat{g}$ be a lift of g. Then*

$$\mathrm{qdim}_{V_{L_g}} V_{\lambda+L_g}[\hat{g}^i] = 1$$

for any $\lambda + L_g \in L_g^*/L_g$ and $i = 0, 1, \ldots, |g| - 1$.

As a corollary, we have

Theorem 4.8. *Let L be an even unimodular lattice. Let g be an element in $O(L)$ and $\hat{g}$ a lift of g. Then the VOA $V_{L_g}^{\hat{g}}$ has group-like fusion, namely, all irreducible modules of $V_{L_g}^{\hat{g}}$ are simple current modules.*

5. Fusion ring of $V_{L_g}^{\hat{g}}$

In this section, we will explain how to determine the group structure and the quadratic space structure for the fusion group of $V_{L_g}^{\hat{g}}$, where L is even unimodular and $g \in O(L)$. The main idea is to use the fusion rules of $V_L^{\phi_g}$. First we recall some facts about the fusion group of $V_L^{\phi_g}$ and the corresponding quadratic space structure from [EMS].

5.1. **Fusion ring of $V_L^{\phi_g}$.** Let L be an even unimodular lattice. Then the lattice VOA V_L is holomorphic. Let n be the order of ϕ_g. Then for each $0 \le i \le n-1$, there is a unique irreducible ϕ_g^i-twisted V_L-module $V_L[\phi_g^i]$. The group $\langle \phi_g \rangle$ acts naturally on $V_L[\phi_g^i]$ and such an action is unique up to a multiplication of an n-th root of unity. Let φ_i be a representation of $\langle \phi_g \rangle$ on $V_L[\phi_g^i]$. Denote

$$W^{i,j} = \{w \in V_L[\phi_g^i] \mid \varphi_i(\phi_g)x = e^{2\pi\sqrt{-1}j/n}x\}$$

for $i, j \in \{0, \dots, n-1\}$. In [EMS], it is proved that the orbifold VOA $V_L^{\phi_g}$ has group-like fusion. Moreover, one can choose the representations φ_i, $i = 0, \dots, n-1$, such that the fusion product

$$W^{i,j} \boxtimes W^{k,\ell} = W^{i+j,k+\ell+c_d(i,j)}, \tag{5.1}$$

where c_d is defined by

$$c_d(i,j) = \begin{cases} 0 & \text{if } i+j < n, \\ d & \text{if } i+j \ge n \end{cases}$$

for $i, j \in \{0, \dots, n-1\}$ and d is determined by the conformal weight ρ of the irreducible twisted module $V_L[\phi_g]$. More precisely, $d = 2n^2\rho \mod n$.

It was also proved that the conformal weight of $W^{i,j}$ defines a quadratic form

$$q(i,j) \equiv \frac{ij}{n} + \frac{i^2 t}{n^2} \mod \mathbb{Z} \tag{5.2}$$

where $t \in \{0, 1, \dots, n-1\}$ and $t = n^2\rho \mod n$. In particular, $d = 2t \mod n$.

In this case, the fusion algebra of $V_L^{\phi_g}$ is isomorphic to the group algebra $\mathbb{C}[D]$, where D is an abelian group defined by a central extension

$$1 \to \mathbb{Z}_n \to D \to \mathbb{Z}_n \to 1$$

associated with the commutator map c_d. The abelian group D is isomorphic to $\mathbb{Z}_{n^2/(n,d)} \times \mathbb{Z}_{(n,d)}$, where (n,d) denotes the gcd of n and d. Notice that q also induces a quadratic form on D.

5.2. **Fusion ring of $V_{L_g}^{\hat{g}}$.** First we recall that there is an isomorphism of the discriminant groups $f : \mathcal{D}(L^g) \to \mathcal{D}(L_g)$ such that

$$V_L = \bigoplus_{\lambda \in (L^g)^*/L^g} V_{\lambda+L^g} \otimes V_{f(\lambda)+L_g}.$$

Since ϕ_g stabilizes the subVOA V_{L_g}, $\hat{g} = \phi_g|_{V_{L_g}}$ defines an automorphism in $\mathrm{Aut}\,(V_{L_g})$. Then $\hat{g}$ is a lift of g in $\mathrm{Aut}\,(V_{L_g})$. Since g fixes all cosets in L_g^*/L_g, $\hat{g}$ also induces an action on $V_{\lambda+L_g}$ for all $\lambda + L_g \in L_g^*/L_g$.

By Lemma 3.5, ϕ_g has order p if p is odd. When p is even, ϕ_g has order p if $\langle x, g^{p/2}(x)\rangle \in 2\mathbb{Z}$ for all $x \in L$ and ϕ_g has order $2p$ if $\langle x, g^{p/2}(x)\rangle \in 1 + 2\mathbb{Z}$ for some $x \in L$.

5.2.1. ***Case 1:*** $|\phi_g| = p$. Suppose ϕ_g has order p. Then, for each $0 \le i \le p-1$, we have

$$V_L[\phi_g^i] = \bigoplus_{\lambda \in (L^g)^*/L^g} V_{\lambda + L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^i].$$

For each ϕ_g-invariant subspace M and $0 \le j < p-1$, we denote

$$M(j) = \{w \in M \mid \phi_g(w) = e^{2\pi\sqrt{-1}j/p} w\}.$$

Therefore, we have

$$V_L^{\phi_g} = \bigoplus_{\lambda \in (L^g)^*/L^g} (V_{\lambda + L^g} \otimes V_{f(\lambda)+L_g})^{\phi_g}$$

and

$$W^{i,j} = V_L[\phi_g^i](j) = \bigoplus_{\lambda \in (L^g)^*/L^g} (V_{\lambda + L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^i])(j). \tag{5.3}$$

By adjusting the action of $\hat{g}$ on $V_{\lambda+L_g}[\hat{g}^i]$ if necessary, we may assume

$$(V_{\lambda + L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^i])(j) \cong V_{\lambda + L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^i](j).$$

Define $I : R(V_L^{\phi_g}) \to R(V_{L_g}^{\hat{g}})$ such that $I(W^{i,j}) = V_{L_g}[\hat{g}^i](j)$ and define $\varphi : (L^g)^*/L^g \times R(V_L^{\phi_g}) \to R(V_{L_g}^{\hat{g}})$ by

$$\varphi(\lambda + L^g, W^{i,j}) = V_{f(\lambda)+L_g}^{\hat{g}} \boxtimes_{V_{L_g}^{\hat{g}}} I(W^{i,j}).$$

Note that the map I is an injective group homomorphism by Formulas (5.1) and (5.3).

Theorem 5.1. *Suppose ϕ_g has order p. Then we have*

$$R(V_{L_g}^{\hat{g}}) \cong (L^g)^*/L^g \times R(V_L^{\phi_g})$$

as an abelian group.

Recall that

$$q(I(W^{i,j})) = q(V_{L_g}[\hat{g}^i])(j)) = q(W^{i,j}) \equiv \frac{ij}{p} + \frac{i^2 t}{p^2} \mod \mathbb{Z},$$

where $t \in \{0, 1, \ldots, p-1\}$ and $t = p^2\rho \mod p$. Hence, $V_{f(\lambda)+L_g}[\hat{g}^i](j)$ has the conformal weight

$$\frac{ij}{p} + \frac{i^2 t}{p^2} - \frac{\langle \lambda, \lambda\rangle}{2} \mod \mathbb{Z}.$$

For $(\lambda+L^g, W) \in (L^g)^*/L^g \times R(V_L^{\phi_g})$, we have $\varphi(\alpha+L^g, W) = V_{f(\lambda)+L_g}^{\hat{g}} \boxtimes_{V_{L_g}^{\hat{g}}} I(W)$. Then the conformal weight of $\varphi(\lambda + L^g, I(W))$ is given by

$$-\frac{\langle\lambda,\lambda\rangle}{2} + q(I(W)) \qquad \text{mod } \mathbb{Z}.$$

5.2.2. ***Case 2:*** $|\phi_g| = 2p$. Let p be even and let $X = \{\alpha \in L \mid \langle\alpha, g^{p/2}(\alpha)\rangle \in 2\mathbb{Z}\}$. If ϕ_g has order $2p$, then $[L : X] = 2$ and $L = X \cup (u + X)$ for some $u \in L \setminus X$.

In this case, $\phi_g^p = \sigma_h$ for some $h \in L/2$ and $h \in X^*$. By definition, ϕ_g fixes V_{L^g} pointwise. Moreover, ϕ_g^p acts trivially on V_{L_g}. Therefore, without loss, we may assume $h \in (L^g)^*$.

Since $\phi_g^p = \sigma_h$ is an inner automorphism, the irreducible ϕ_g^p-twisted module $V_L[\phi_g^p]$ is given by $V_L^{(h)} \cong V_{h+L}$. Recall that

$$V_L = \bigoplus_{\lambda\in(L^g)^*/L^g} V_{\lambda+L^g} \otimes V_{f(\lambda)+L_g}.$$

Therefore, we have

$$V_L[\phi_g^p] = V_{h+L} = \bigoplus_{\lambda\in(L^g)^*/L^g} V_{h+\lambda+L^g} \otimes V_{f(\lambda)+L_g}.$$

Moreover, for each $0 \le i \le p-1$, we have

$$V_L[\phi_g^i] = \bigoplus_{\lambda\in(L^g)^*/L^g} V_{\lambda+L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^i],$$
$$V_L[\phi_g^{p+i}] = \bigoplus_{\lambda\in(L^g)^*/L^g} V_{h+\lambda+L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^i].$$

In this case, we have

$$V_L^{\phi_g} = \bigoplus_{\lambda\in X'/L^g} (V_{\lambda+L^g} \otimes V_{f(\lambda)+L_g})^{\phi_g},$$

where X' is the image of X under the natural projection from L to $(L^g)^*$.

We will fix $u \in L \setminus X$ such that the order of $h + u' + L^g$ in $\mathcal{D}(L^g)$ is minimal, where u' denotes the image of u under the natural projection from L to $(L^g)^*$. Notice that $h + u + L^g$ has either order 1 or 2. If $h \notin X'$, then $h + u' \in X'$. We may choose $u' = -h$ in this case.

Recall that $q(W^{i,j}) \equiv ij/2p + i^2t/4p^2 \mod \mathbb{Z}$, where $t = 4p^2\rho \mod 2p$ and $t \in \{0, 1, \ldots, 2p-1\}$. Notice that ρ is also the conformal weight of $V_{f(\lambda)+L_g}[\hat{g}]$ for any $\lambda \in (L^g)^*$ and $\hat{g} = \phi_g|_{V_{L_g}}$ has order p. Thus, the weights of $V_{\lambda+L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}])$ are in $\rho + \frac{\langle\lambda,\lambda\rangle}{2} + \frac{1}{p}\mathbb{Z}$.

By [EMS], $\rho - \frac{t}{4p^2} \in \frac{1}{2p}\mathbb{Z}$. There are also two cases:

Case a: $\rho - \frac{t}{4p^2} \in \frac{1}{p}\mathbb{Z}$.

In this case, the weights of $V_{L^g} \otimes V_{L_g}[\hat{g}]$ are in $\frac{t}{4p^2} + \frac{1}{p}\mathbb{Z}$ and we have

(5.4)

$$W^{i,j} = \begin{cases} \bigoplus_{\lambda \in X'/L^g} (V_{\lambda+L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^i])(j), & \text{if } j \text{ is even and } 0 \le i < p, \\ \bigoplus_{\lambda \in X'/L^g} (V_{\lambda+u'+L^g} \otimes V_{f(\lambda+u')+L_g}[\hat{g}^i])(j), & \text{if } j \text{ is odd and } 0 \le i < p, \\ \bigoplus_{\lambda \in X'/L^g} (V_{h+\lambda+L^g} \otimes V_{f(\lambda)+L_g}[\hat{g}^{i-p}])(j), & \text{if } j \text{ is even and } p \le i < 2p, \\ \bigoplus_{\lambda \in X'/L^g} (V_{h+\lambda+u'+L^g} \otimes V_{f(\lambda+u')+L_g}[\hat{g}^{i-p}])(j), & \text{if } j \text{ is odd and } p \le i < 2p. \end{cases}$$

By adjusting the action of $\hat{g}$ on $V_{\lambda+L_g}[\hat{g}^i]$ if necessary, we may assume

$$(V_{L^g} \otimes V_{L_g}[\hat{g}^i])(j) = V_{L^g} \otimes V_{L_g}[\hat{g}^i](\frac{j}{2}), \quad \text{if } j \text{ is even and } 0 \le i < p,$$

$$(V_{u'+L^g} \otimes V_{f(u')+L_g}[\hat{g}^i])(j) = V_{u'+L^g} \otimes V_{f(u')+L_g}[\hat{g}^i](\frac{j-1}{2}), \quad \text{if } j \text{ is odd and } 0 \le i < p,$$

$$(V_{h+L^g} \otimes V_{L_g}[\hat{g}^i])(j) = V_{h+L^g} \otimes V_{L_g}[\hat{g}^i](\frac{j}{2}), \quad \text{if } j \text{ is even and } p \le i < 2p,$$

$$(V_{h+u'+L^g} \otimes V_{f(u')+L_g}[\hat{g}^i])(j) = V_{h+u'+L^g} \otimes V_{f(u')+L_g}[\hat{g}^i](\frac{j-1}{2}), \quad \text{if } j \text{ is odd and } p \le i < 2p.$$

Define $I : R(V_L^{\phi_g}) \to R(V_{L_g}^{\hat{g}})$ such that

$$I(W^{i,j}) = \begin{cases} V_{f(u')+L_g}[\hat{g}^i](\frac{j-1}{2}) & \text{if } 0 \le i < p,\ j \text{ odd}, \\ V_{L_g}[\hat{g}^i](\frac{j}{2}) & \text{if } 0 \le i < p,\ j \text{ even}, \\ V_{f(h+u')+L_g}[\hat{g}^{i-p}](\frac{j-1}{2}) & \text{if } p \le i < 2p,\ j \text{ odd}, \\ V_{f(h)+L_g}[\hat{g}^{i-p}](\frac{j}{2}) & \text{if } p \le i < 2p,\ j \text{ even}. \end{cases}$$

Notice that the map I may depend on the choice of u'.

Lemma 5.2. *The map I is a group homomorphism. Moreover, I is 1 to 1 if $h \in X'$; otherwise, I is 2 to 1.*

Let $Y' = \{a \in (L^g)^* \mid \langle a, h\rangle \in \mathbb{Z} \text{ and } \langle a, u'\rangle \in \mathbb{Z}\}$. Then $X' > Y' > L^g$. Note that $h, u, \notin Y'$ since $\langle h, u'\rangle \notin \mathbb{Z}$, and $Y'/L^g \times H \cong (L^g)^*/L^g$, where H is the subgroup of $(L^g)^*/L^g$ generated by $h + L^g$ and $u' + L^g$.

Note that $X' = Y'$ if $h \notin X'$, i.e., $\langle h, h\rangle \notin \mathbb{Z}$ or $u' + h \in X'$; otherwise, we have $[(L^g)^* : Y'] = 2^2$.

Now define $\varphi : Y'/L^g \times I(R(V_L^{\phi_g})) \to R(V_{L_g}^{\hat{g}})$ by

$$\varphi(\lambda + L^g, I(W^{i,j})) = V_{f(\lambda)+L_g}^{\hat{g}} \boxtimes_{V_{L_g}^{\hat{g}}} I(W^{i,j}).$$

Then

$$\varphi(\lambda+L^g, I(W^{i,j})) = \begin{cases} V_{f(\lambda+u')+L_g}[\hat{g}^i](\frac{j-1}{2}) & \text{if } 0\le i<p,\ j \text{ odd},\\ V_{f(\lambda)+L_g}[\hat{g}^i](\frac{j}{2}) & \text{if } 0\le i<p,\ j \text{ even},\\ V_{f(\lambda+h+u')+L_g}[\hat{g}^{i-p}](\frac{j-1}{2}) & \text{if } p\le i<2p,\ j \text{ odd},\\ V_{f(\lambda+h)+L_g}[\hat{g}^{i-p}](\frac{j}{2}) & \text{if } p\le i<2p,\ j \text{ even}. \end{cases} \tag{5.5}$$

Theorem 5.3. *The map φ is an isomorphism of groups and we have*

$$R(V_{L_g}^{\hat{g}}) \cong Y'/L^g \times I(R(V_L^{\phi_g})).$$

For the quadratic form, we have

$$q(I(W^{i,j})) = \begin{cases} \frac{ij}{n} + \frac{i^2t}{n^2} \mod \mathbb{Z} & \text{if } j \text{ is even},\\ \frac{ij}{n} + \frac{i^2t}{n^2} - \frac{\langle u',u'\rangle}{2} \mod \mathbb{Z} & \text{if } j \text{ is odd}, \end{cases} \tag{5.6}$$

For $(\lambda+L^g, I(W)) \in Y'/L^g\times I(R(V_L^{\phi_g}))$, we have $\varphi(\lambda+L^g, I(W)) = V_{f(\lambda)+L_g}^{\hat{g}} \boxtimes_{V_{L_g}^{\hat{g}}} I(W)$. Then the conformal weight of $\varphi(\lambda+L^g, I(W))$ is given by

$$-\frac{\langle\lambda,\lambda\rangle}{2} + q(I(W)) \mod \mathbb{Z}$$

by (5.5) and (5.6).

Case b: $\rho - \frac{t}{4p^2} \in \frac{1}{2p}\mathbb{Z}\setminus\frac{1}{p}\mathbb{Z}$.

In this case, the weights of $V_{L^g}\otimes V_{L_g}[\hat{g}]$ are in $\frac{t}{4p^2}+\frac{1}{2p}\mathbb{Z}\setminus\frac{1}{p}\mathbb{Z}$ but the weights of $V_{u'+L^g}\otimes V_{f(u')+L_g}[\hat{g}]$ are in $\frac{t}{4p^2}+\frac{1}{p}\mathbb{Z}$. Then

$$W^{1,0} = \bigoplus_{\lambda\in X'/L^g} (V_{\lambda+u'+L^g}\otimes V_{f(\lambda+u')+L_g}[\hat{g}])(0);$$

notice that $q(W^{1,0}) = t/4p^2 \mod \mathbb{Z}$. Similarly, we also have

$$W^{1,j} = \bigoplus_{\lambda\in X'/L^g} (V_{\lambda+(\bar{j}+1)u'+L^g}\otimes V_{f(\lambda+(\bar{j}+1)u')+L_g}[\hat{g}])(j).$$

By the fusion rules, we also have

$$W^{i,0} = \begin{cases} \bigoplus_{\lambda\in X'/L^g} (V_{\lambda+\bar{i}u'+L^g}\otimes V_{f(\lambda+\bar{i}u')+L_g}[\hat{g}^i])(0) & \text{if } 0\le i<p,\\ \bigoplus_{\lambda\in X'/L^g} (V_{h+\lambda+\bar{i}u'+L^g}\otimes V_{f(\lambda+\bar{i}u')+L_g}[\hat{g}^i])(0) & \text{if } p\le i<2p. \end{cases}$$

and

$$W^{i,j} = \begin{cases} \bigoplus_{\lambda\in X'/L^g} (V_{\lambda+(\bar{i}+\bar{j})u'+L^g}\otimes V_{f(\lambda+(\bar{i}+\bar{j})u')+L_g}[\hat{g}^i])(j), & \text{if } 0\le i<p,\\ \bigoplus_{\lambda\in X'/L^g} (V_{h+(\bar{i}+\bar{j})u'+\lambda+L^g}\otimes V_{f(\lambda+(\bar{i}+\bar{j})u')+L_g}[\hat{g}^{i-p}])(j), & \text{if } p\le i<2p. \end{cases} \tag{5.7}$$

By adjusting the action of $\hat{g}$ on $V_{\lambda+L_g}[\hat{g}^i]$, we may also assume

$$(V_{L^g}\otimes V_{L_g}[\hat{g}^i])(j) = V_{L^g}\otimes V_{L_g}[\hat{g}^i](\left\lfloor\frac{j}{2}\right\rfloor), \quad \text{if } i+j \text{ is even and } 0\le i<p,$$

$$(V_{u'+L^g}\otimes V_{f(u')+L_g}[\hat{g}^i])(j) = V_{u'+L^g}\otimes V_{f(u')+L_g}[\hat{g}^i](\left\lfloor\frac{j}{2}\right\rfloor), \quad \text{if } i+j \text{ is odd and } 0\le i<p,$$

$$(V_{h+L^g}\otimes V_{L_g}[\hat{g}^i])(j) = V_{h+L^g}\otimes V_{L_g}[\hat{g}^i](\left\lfloor\frac{j}{2}\right\rfloor), \quad \text{if } i+j \text{ is even and } p\le i<2p,$$

$$(V_{h+u'+L^g}\otimes V_{f(u')+L_g}[\hat{g}^i])(j) = V_{h+u'+L^g}\otimes V_{f(u')+L_g}[\hat{g}^i](\left\lfloor\frac{j}{2}\right\rfloor), \quad \text{if } i+j \text{ is odd and } p\le i<2p,$$

where $\lfloor x\rfloor$ denotes the greatest integer that is less than or equal to x.

Define $I: R(V_L^{\phi_g}) \to R(V_{L_g}^{\hat{g}})$ such that

$$I(W^{i,j}) = \begin{cases} V_{L_g}[\hat{g}^i](\lfloor\frac{j}{2}\rfloor) & \text{if } 0\le i<p,\ i+j \text{ even },\\ V_{f(u')+L_g}[\hat{g}^i](\lfloor\frac{j}{2}\rfloor) & \text{if } 0\le i<p,\ i+j \text{ odd},\\ V_{f(h)+L_g}[\hat{g}^{i-p}](\lfloor\frac{j}{2}\rfloor) & \text{if } p\le i<2p,\ i+j \text{ even },\\ V_{f(h+u')+L_g}[\hat{g}^{i-p}](\lfloor\frac{j}{2}\rfloor) & \text{if } p\le i<2p,\ i+j \text{ odd}. \end{cases}$$

By the same arguments as in last section, we have the following results.

Lemma 5.4. *The map I is a group homomorphism. Moreover, I is 1 to 1 if $h\in X'$; otherwise, I is 2 to 1.*

Let $Y' = \{a\in (L^g)^* \mid \langle a,h\rangle \in \mathbb{Z} \text{ and } \langle a,u'\rangle\in\mathbb{Z}\}$ and H the subgroup of $(L^g)^*/L^g$ generated by $h+L^g$ and $u'+L^g$. Then $Y'/L^g\times H\cong (L^g)^*/L^g$.

Theorem 5.5. *As an abelian group, we have*

$$R(V_{L_g}^{\hat{g}}) \cong Y'/L^g\times I(R(V_L^{\phi_g})).$$

For the quadratic form, we have

(5.8)
$$q(W^{i,j}) = \begin{cases} q(V_{L_g}[\hat{g}^i](\lfloor\frac{j}{2}\rfloor)), & \text{if } i+j \text{ is even and } 0\le i<p,\\ q(V_{u'+L^g}) + q(V_{f(u')+L_g}[\hat{g}^i](\lfloor\frac{j}{2}\rfloor), & \text{if } i+j \text{ is odd and } 0\le i<p,\\ q(V_{f(h)+L_g}[\hat{g}^{i-p}](\lfloor\frac{j}{2}\rfloor)), & \text{if } i+j \text{ is even and } p\le i<2p,\\ q(V_{u'+L^g}) + q(V_{f(h+u')+L_g}[\hat{g}^{i-p}](\lfloor\frac{j}{2}\rfloor)), & \text{if } i+j \text{ is odd and } p\le i<2p, \end{cases}$$

by (5.4). Hence,

$$q(I(W^{i,j})) = \begin{cases} \frac{ij}{n}+\frac{i^2t}{n^2} \mod \mathbb{Z} & \text{if } i+j \text{ is even},\\ \frac{ij}{n}+\frac{i^2t}{n^2}-\frac{\langle u',u'\rangle}{2} \mod \mathbb{Z} & \text{if } i+j \text{ is odd}, \end{cases} \tag{5.9}$$

For $(\lambda + L^g, I(W)) \in Y'/L^g \times I(R(V_L^{\phi_g}))$, the conformal weight of $V_{f(\lambda)+L_g}^{\hat{g}} \boxtimes_{V_{L_g}^{\hat{g}}} I(W)$ is given by

$$-\frac{\langle \lambda, \lambda \rangle}{2} + q(I(W)) \mod \mathbb{Z}$$

6. Leech lattice and some explicit examples

Next we will study several explicit examples associated with the Leech lattice. Recall that the cases for prime order elements have been treated in [LS17] (see also [HS14]). In this section, we will discuss the cases as listed in Table 1. In [Hö], a construction of holomorphic vertex operator algebras of central charge 24 has been proposed by using some orbifold VOA associated with coinvariant lattices of the Leech lattice. In particular, the cases in Table 1 have been discussed.

Table 1: Standard lift of $g \in O(\Lambda)$

$\lvert g\rvert$	$rank(\Lambda^g)$	Conjugacy class	Cycle shape	$\lvert\phi_g\rvert$	ρ
4	10	$4C$	$1^4 2^2 4^4$	4	3/4
6	6	$6G$	$2^3 6^3$	12	11/12
6	8	$6E$	$1^2 2^2 3^2 6^2$	6	5/6
8	6	$8E$	$1^2 2^1 4^1 8^2$	8	7/8
10	4	$10F$	$2^2 10^2$	20	19/20

By Magma, it is easy to verify that

$$\Lambda_g = (1-g)\Lambda = (1-g)\Lambda_g^*$$

for an isometry g listed in Table 1. Hence, $|\Lambda_g^*/\Lambda_g| = \det(1-g)$ by Lemma 4.3. In fact, $\mathcal{D}(\Lambda_g)$ is determined by the cycle shape of g for these cases.

First, we discuss the cases which $|\phi_g| = |g|$, i.e., $4C, 6E$ and $8E$.

6.1. **Conjugacy class $4C$.** Let g be an isometry of conjugacy class $4C$ in $O(\Lambda)$. Then g^2 is in the conjugacy class $2A$. In this case, the fixed point sublattice of g has rank 10 and $\mathcal{D}(\Lambda^g) \cong 2^2 \times 4^4$. Moreover, $(\mathcal{D}(\Lambda^g), q)$ is a quadratic space of the type $2^{2,+} \times 4^{4,+}$.

For $4^{2n,\epsilon}$, we mean an quadratic space (X, q) over $\mathbb{Z}_4$ such that $(X/2X, \bar{q})$ is a non-singular quadratic space of type $2^{2n,\epsilon}$ over $\mathbb{Z}_2$. Notice that $2X \cong 2^{2n}$ and is totally singular with respect to the quadratic q.

Since $V_\Lambda[\phi_g]$ has the conformal weight 3/4, we have $t \equiv 0 \mod 4$; thus, $R(V_\Lambda^{\phi_g}) \cong 4^2$. Moreover, $(I(R(V_\Lambda^{\phi_g})), q) \cong 4^{2,+}$ by (5.3). Therefore,

$$R(V_{\Lambda_g}^{\hat{g}}) \cong \mathcal{D}(\Lambda^g) \times R(V_\Lambda^{\phi_g}) \cong 2^2 \times 4^6$$

by Theorem 5.1 and $(R(V_{\Lambda_g}^{\hat{g}}),q)$ is a quadratic space of type $2^{2,+}\times 4^{6,+}$ by the discussions in Section 5.2.1.

6.2. **Conjugacy class** $6E$**.** Let g be an isometry of conjugacy class $6E$ in $O(\Lambda)$. Then g^2 is in the conjugacy class $3A$ and g^3 is in the conjugacy class $2A$. The fixed point sublattice of g has rank 8 and is isometric to $A_2\otimes D_4$. Moreover, the discriminant form $(\mathcal{D}(\Lambda^g),q)\cong 2^{4,+}\times 3^{4,+}$ (cf. [GL13]). Since the conformal weight of $V_\Lambda[\phi_g]$ is 5/6, we have $t\equiv 0 \mod 6$ and thus $R(V_\Lambda^{\phi_g})\cong 6^2$. By (5.2), it is easy to verify that the quadratic form associated with $(R(V_\Lambda^{\phi_g}),q)$ is isometric to $2^{2,+}\times 3^{2,+}$. Thus, $R(V_{\Lambda_g}^{\hat{g}})$ is isometric to a quadratic space of type $2^{6,+}\times 3^{6,+}$.

6.3. **Conjugacy class** $8E$**.** Let g be an isometry of conjugacy class $4C$ in $O(\Lambda)$. Then g^2 is in the conjugacy class $4C$ and g^4 is in the class $2A$. The fixed point sublattice of g has rank 8 and $\mathcal{D}(\Lambda^g)\cong 2\times 4\times 8^2$. By using computer, one can also show that the discriminant form $(\mathcal{D}(\Lambda^g),q_L)$ has the type $2^+\cdot 4^+\cdot 8^{2,+}$. Again we have $t=0 \mod 8$ in this case. Therefore, $R(V_{\Lambda_g}^{\hat{g}})\cong 2\times 4\times 8^4$ and $(R(V_{\Lambda_g}^{\hat{g}}),q)\cong 2^+\times 4^+\times 8^{4,+}$.

Next we discuss the cases that $|\phi_g|=2|g|$.

6.4. **Conjugacy class** $6G$**.** Let g be an isometry of conjugacy class $6G$ in $O(\Lambda)$. Then g^2 is in the conjugacy class $3B$ and g^3 is in the class $2C$. In this case, the fixed point lattice has rank 6 and the discriminant form $(\mathcal{D}(\Lambda^g),q)$ is isometric to $2^{6,-}\times 3^3$. In this case, $|\phi_g|$ has order 12 and $\phi_g^6=\sigma_h$ for some $h\in\frac{1}{2}\Lambda_g$ and $\langle h,h\rangle=2$. Moreover, we can choose u' such that $\langle u',u'\rangle=\frac{3}{2}$. The irreducible ϕ_g-twisted module has the conformal weight $\rho=11/12$; hence we have $t\equiv 0 \mod 12$ and $\rho\notin\frac{1}{6}\mathbb{Z}$. That means we have Case 2b as described in Section 5.

Since $\langle h,h\rangle\in\mathbb{Z}$, we have $R(V_{\Lambda_g}^{\hat{g}})\cong 2^4\times 4^2\times 3^5$ as an abelian group by Theorem 5.5. It is also easy to verify that $(Y'/\Lambda^g,q)\cong 2^{4,+}\times 3^3$ as a quadratic space, where $Y'=\{a\in(\Lambda^g)^*\mid \langle a,h\rangle\in\mathbb{Z} \text{ and } \langle a,u'\rangle\in\mathbb{Z}\}$.

By (5.9), we also have

$$q(I(W^{i,j}))=\frac{ij}{12}-\frac{3}{4}(\bar{i}+\bar{j}) \mod \mathbb{Z}.$$

Then

$$(I(R(V_\Lambda^{\phi_g})),q)\cong 4^{2,-}\times 3^{2,+}.$$

Hence, $(R(V_{\Lambda_g}^{\hat{g}}),q)\cong 2^{4,+}\times 4^{2,-}\times 3^5$.

6.5. **Conjugacy class** $10F$**.** Let g be an isometry of conjugacy class $10F$ in $O(\Lambda)$. Then g^2 is in the conjugacy class $5B$ and g^5 is in the class $2C$. The fixed point sublattice has rank 4 and the discriminant form $(\mathcal{D}(\Lambda^g), q)$ is isometric to $2^{4,+} \times 5^{2,+}$. In this case, $|\phi_g|$ has order 20 and $\phi_g^{10} = \sigma_h$ for some $h \in \frac{1}{2}\Lambda_g$ and $\langle h, h\rangle = 2$. We can also choose u' such that $\langle u', u'\rangle = 3/2$.

Again, we have $t \equiv 0 \mod 20$ and $\langle h, h\rangle \in \mathbb{Z}$ in this case; thus, we have $R(V_{\Lambda_g}^{\hat{g}}) \cong 2^2 \times 4^2 \times 5^4$ as an abelian group by Theorem 5.3.

Let $Y' = \{a \in (\Lambda^g)^* \mid \langle a, h\rangle \in \mathbb{Z} \text{ and } \langle a, u'\rangle \in \mathbb{Z}\}$. Then it can be verified that $(Y'/\Lambda^g, q) \cong 2^{2,-} \times 5^{2,+}$. Also, by (5.9), we have

$$q(I(W^{i,j})) = \frac{ij}{20} - \frac{3}{4}(\bar{i} + \bar{j}) \mod \mathbb{Z}.$$

Thus, $I(R(V_\Lambda^{\phi_g})), q) \cong 4^{2,-} \times 5^{2,+}$ and $(R(V_{\Lambda_g}^{\hat{g}}), q)$ has the type $2^{2,-} \times 4^{2,-} \times 5^{4,+}$.

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